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Letter to the Editor

# A new approach for solving a complex-valued differential equation

L. Cveticanin*

*Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovica 6, 21000 Novi Sad, Serbia and Montenegro*

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**Abstract**

In this paper two analytical approximate solving procedures for a complex-valued differential equation are developed. One of the methods represents the generalization of the Krylov–Bogolubov method for a strong differential equation with complex function. The second method is based on the first integrals of the system. The differential equation is transformed introducing the perturbed first integrals and the polar co-ordinates. The solution is obtained applying the straightforward series expansion. The solution for the special case of without impact initial conditions is considered. The method is applied on the system with strong cubic non-linearity. The small gyroscopic function and damping function are introduced. The analytical approximate solution is compared with numerical exact one and shows a good agreement.


## 1. Introduction

The differential equation considered in this paper is

$$\ddot{z} + c_1 z + zF(z\bar{z}) = \varepsilon Z(z, \dot{z}, cc), \tag{1}$$

where $z$ is a complex function, $\mathrm{i} = \sqrt{-1}$ is an imaginary unit, $\bar{z}$ is a complex conjugate function, $\varepsilon$ is a small parameter, $Z$ is a complex function, $F$ is a function of $(z\bar{z})$ and $c_1$ is a constant coefficient. The differential equation (1) is an ordinary second order differential equation with a complex function where $c_1 z$ is the linear term, $zF(z\bar{z})$ is the strong non-linear term whose order of non-linearity depends on the degree of $z\bar{z}$, and $\varepsilon Z$ is a small function which depends on complex function $z$, its time derivative $\dot{z}$ and complex conjugate functions $cc$. The general initial conditions are

$$z(0) = z_0, \quad \dot{z}(0) = \dot{z}_0. \tag{2}$$

*Tel.: +381-21-350-413; fax: +381-21-58-133.
*E-mail address:* cveticanin@uns.ns.ac.yu (L. Cveticanin).

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The differential equation (1) with the initial conditions (2) represents the extension of the previously investigated differential equation with small non-linearity

$$\ddot{z} + c_1 z = \varepsilon Z(z, \dot{z}, cc), \tag{3}$$

The differential equation (3) is widely investigated and approximate analytical solution methods [1] and [2], which are based on the Krylov–Bogolubov [3] and the Bogolubov–Mitropolski [4] procedures, are developed. Mahmoud [5] introduced the generalized averaging solution method for the differential equation (3) with a complex function and small non-linearity of a cubic type. Unfortunately, the suggested procedures in this form are not applicable for solving the strong non-linear differential equation (1).

There are many papers dealing with solution methods for strong non-linear differential equation which describes the one-degree-of-freedom system. Most of the procedures are of the perturbation type [6–12]: the elliptic-Krylov–Bogolubov method, the multiple scale method, etc. The harmonic balance method is also applied to strong non-linear systems [13,14]. The averaging method for the solution in the form of elliptic functions is developed by Coppola and Rand [15] and Belhaq and Lakrad [16]. Xu and Cheng [17] developed the solution in the form of generalized harmonic functions where the argument is an implicit non-linear time function.

Based on those aforementioned methods the analytical solution procedures for the two coupled strong non-linear differential equations are considered in Refs. [18–24]. The special group of two-degrees-of-freedom systems described with a strong non-linear differential equation with complex function is investigated in Refs. [25–28]. The analytic solution for the 'natural' complex valued version is described in Ref. [29]. The non-linearity is of the cubic type. Such differential equations are also considered in Refs. [30–33].

In this paper the generalization of the solution procedure to the complex-valued non-linear differential equation (1) with initial conditions (2) is carried out. Two solution procedures are developed. The first method represents the generalization of the Krylov–Bogolubov procedure for a complex valued differential equation where the trial perturbed solution of the system is based on the closed form solution of the generating equation ($\varepsilon = 0$) which is assumed to be known and denoted as in the previously mentioned papers in terms of elliptic functions or generalized harmonic functions. The other method is based on the first integrals of the generating equation ($\varepsilon = 0$). The differential equations are transformed into the new variables based on these first integrals. Introducing the power series expansion and equating the terms of the same order sets of first order differential equations are obtained. Solving the equations the solution of the higher order approximation is denoted. The method is tested on the examples where the strong non-linearity is of the cubic type and small gyroscopic and damping terms exist. The obtained results are compared with exact numerical one and discussed.

## 2. Generalization of the Krylov–Bogolubov method for the differential equation with complex function

The method represents a generalization of the Krylov–Bogolubov method developed for the complex-valued differential equation (3) with small non-linearity [1]. The method is based on the generating solution of the differential equation (1) for $\varepsilon = 0$ with arbitrary initial conditions (2).

Perturbing that solution, the trial solution of Eq. (1) is formed. Using this assumed solution and some constraints the differential equation (1) is transformed into a new system of two coupled first order complex-valued differential equations. The approximate solution is obtained using the averaging procedure.

For $\varepsilon = 0$ the generating equation for (1) has the form

$$\ddot{z} + c_1 z + zF(z\bar{z}) = 0. \tag{4}$$

Its solution is

$$z = f(K_1, K_2, K_3, K_4, t), \tag{5}$$

where $K_1, K_2, K_3, K_4$ are constants dependent on the initial conditions (2). Perturbing this generating solution (5) the trial solution of Eq. (1) is

$$z = f(K_1(t), K_2(t), K_3(t), K_4(t), t), \tag{6}$$

where $K_1(t), K_2(t), K_3(t), K_4(t)$ are real time dependent functions. The solution (6) has to satisfy the equation (1). It requires the following constraint to be introduced: the first time derivative of (6) must have the same form as the first time derivative of the generating solution (5) and it is

$$\dot{z} = \frac{\partial f}{\partial t}. \tag{7}$$

Comparing the time derivative of (6) and the relation (7) the following complex-valued equation is obtained

$$\sum_{j=1}^{j=4} \frac{\partial f}{\partial K_j} \dot{K}_j = 0. \tag{8}$$

It is a first order ordinary differential equation with real variables but with real and imaginary terms.

Introducing the solution (6), the relation (7) and its time derivative into (1) it is

$$\sum_{j=1}^{j=4} \frac{\partial}{\partial K_j}\left(\frac{\partial f}{\partial t}\right) \dot{K}_j = \varepsilon Z\left(f, \frac{\partial f}{\partial t}, cc\right). \tag{9}$$

Eq. (9) is also a first order differential equation with real and imaginary terms which with Eq. (8) represent the transformed differential equation (1) with the new variables $K_j$. Separating the real and imaginary terms in the differential equations (8) and (9) a system of four coupled first order differential equations is obtained

$$\dot{K}_j = \frac{\Delta_{Kj}}{\Delta}, \quad j = 1, \ldots, 4, \tag{10}$$

where

$$\Delta = |a_{ij}|, \quad i = 1, \ldots, 4, \; j = 1, \ldots, 4,$$

$$a_{1j} = \mathrm{Re}\left(\frac{\partial f}{\partial K_j}\right), \quad a_{2j} = \mathrm{Im}\left(\frac{\partial f}{\partial K_j}\right),$$

$$a_{3j} = \mathrm{Re}\left(\frac{\partial}{\partial K_j}\left(\frac{\partial f}{\partial t}\right)\right), \quad a_{4j} = \mathrm{Im}\left(\frac{\partial}{\partial K_j}\left(\frac{\partial f}{\partial t}\right)\right), \tag{11}$$

$$\Delta_{Kj} = \begin{vmatrix} b_{1p} \\ b_{2p} \\ b_{3p} \\ b_{4p} \end{vmatrix}, \quad p = 1, \ldots, 4, \tag{12}$$

$$\begin{aligned} b_{1p} &= 0 \quad \text{for } p = j, \qquad b_{1p} = a_{1j} \quad \text{for } p \neq j, \\ b_{2p} &= 0 \quad \text{for } p = j, \qquad b_{2p} = a_{2j} \quad \text{for } p \neq j, \\ b_{3p} &= \varepsilon\,\mathrm{Re}(Z) \quad \text{for } p = j, \qquad b_{3p} = a_{3j} \quad \text{for } p \neq j, \\ b_{4p} &= \varepsilon\,\mathrm{Im}(Z) \quad \text{for } p = j, \qquad b_{4p} = a_{4j} \quad \text{for } p \neq j. \end{aligned} \tag{13}$$

Solving the differential equations (10) for $K_j(t)$ the exact solution (6) of the differential equation (1) is obtained.

Unfortunately, usually it is impossible to find the closed form solution for (10) in analytical form. As it is suggested by Krylov and Bogolubov [3] the equations (10) have to be averaged. Namely, as the motion is periodical with period $T$ the averaged equations are

$$\dot{K}_j = \left\langle \frac{\Delta_{Kj}}{\Delta} \right\rangle, \quad j = 1, \ldots, 4, \tag{14}$$

where $\langle \cdot \rangle \equiv (1/T)\int_0^T (\cdot)\mathrm{d}T$. Solving the averaged system of differential equations (14) and substituting the so obtained values for $K_j(t)$ into (6) the approximate analytical solution of (1) is obtained.

The advantage of the method is its simplicity. The solution method is based on the generating solution which is perturbed. Unfortunately, it is difficult to obtain the closed form generating solution of (4) for arbitrary initial conditions (2). The averaging which is introduced in solution process has a significant influence on the accuracy of the solution. Usually, the approximate averaged solution is on the top of the exact solution only for a short time interval and small values of non-linearity.

## 3. Method based on the first integrals

To eliminate the disadvantage of the previous method the following procedure is suggested. The differential equation (1) is transformed into new variables which represent the polar co-ordinates and the two first integrals of the system. The power series approach is applied. Separating the terms with the same order of the small parameter $\varepsilon$ sets of first order differential equations are obtained. Solving these equations the solution in the first or higher approximation are obtained.

Introducing the polar co-ordinates $\rho$ and $\theta$ where $\rho$ is the position co-ordinate and $\theta$ the angle co-ordinate, and the polar form of the complex function $z = \rho\exp(\mathrm{i}\theta)$ into (1) and (2) and separating the real and imaginary terms a system of two second order differential equations is obtained

$$\ddot{\rho} - \rho\dot{\theta}^2 + c_1\rho + \rho F(\rho) = \varepsilon\Phi_\rho(\rho, \dot{\rho}, \theta, \dot{\theta}),$$
$$\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta} = \varepsilon\Phi_\theta(\rho, \dot{\rho}, \theta, \dot{\theta}), \tag{15}$$

with

$$\rho(0) = A, \quad \dot{\rho}(0) = B, \quad \theta(0) = C, \quad \dot{\theta}(0) = D, \tag{16}$$

where $\varepsilon\Phi_\rho$ and $\varepsilon\Phi_\theta$ are small real functions. The solution method in this paper considers the differential equations (15) and initial conditions (16).

The generating equation, when $\varepsilon = 0$, is

$$\ddot{\rho}_0 - \rho_0\dot{\theta}_0^2 + \rho_0 F(\rho_0) = 0,$$
$$\rho_0\ddot{\theta}_0 + 2\dot{\rho}_0\dot{\theta}_0 = 0. \tag{17}$$

The differential equations (17) have the form of the differential equations of central motion of a particle. Eq. $(17)_2$ represents a cyclic first integral

$$\rho_0^2\dot{\theta}_0 = A^2 D \equiv K_{10} = const., \tag{18}$$

for the cyclic co-ordinate $\theta_0$. This integral represents twice the value of the so called sectorial velocity. Its value is constant for the system (17).

Introducing the relation (18) into $(17)_1$ it transforms to another first integral of energy type

$$\tfrac{1}{2}\dot{\rho}_0^2 + \frac{1}{2}\frac{K_{10}^2}{\rho_0^2} + \tfrac{1}{2}c_1\rho_0^2 + \left(\int \rho F(\rho)\mathrm{d}\rho\right)_0$$
$$= \tfrac{1}{2}B^2 + \frac{1}{2}\frac{K_{10}^2}{A^2} + \tfrac{1}{2}c_1 A^2 + \left(\int \rho F(\rho)\mathrm{d}\rho\right)_A \equiv K_{20} = const. \tag{19}$$

It can be concluded that for the conservative system (17) the energy integral (19) is constant.

Solving the relation

$$\int_A^{\rho_0} \frac{\mathrm{d}\rho_0}{\sqrt{2K_{20} - K_{10}^2/\rho_0^2 - c_1\rho_0^2 - 2\left(\int \rho F(\rho)\mathrm{d}\rho\right)_0}} = t, \tag{20}$$

the $\rho_0 = \rho_0(t)$ co-ordinate variation is obtained. Substituting the solution of (20) into (18) and integrating it is

$$\theta_0 = C + K_{10}^2 \int_0^t \frac{\mathrm{d}t}{\rho_0^2(t)}. \tag{21}$$

Using the results of the generating equation (17) the differential equation (15) is transformed into a system of four coupled first order ordinary differential equations

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{K_1}{\rho^2},$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \sqrt{2K_2 - \frac{K_1^2}{\rho^2} - c_1\rho^2 - 2\int \rho F(\rho)\mathrm{d}\rho},$$

$$\frac{dK_1}{dt} = \varepsilon\rho\Phi_\theta(\rho, \theta, K_1, K_2),$$

$$\frac{dK_2}{dt} = \varepsilon\left(2K_2 - \frac{K_1^2}{\rho^2} - c_1\rho^2 - 2\int \rho F(\rho)d\rho\right)^{1/2}\Phi_\rho(\rho, \theta, K_1, K_2), \tag{22}$$

with initial conditions

$$\rho(0) = A, \quad \theta(0) = C, \quad K_1(0) = K_{10}, \quad K_2(0) = K_{20}, \tag{23}$$

where

$$K_1 = \rho^2\dot{\theta}, \quad K_2 = \tfrac{1}{2}\dot{\rho}^2 + \frac{1}{2}\frac{K_1^2}{\rho^2} + \tfrac{1}{2}c_1\rho^2 + \int \rho F(\rho)d\rho, \tag{24}$$

are the perturbed time dependent first integrals. So the task of obtaining the solution $z(t)$ of Eq. (1) i.e., (15) has been transformed into the equivalent one of obtaining the four solutions $K_1(t)$, $K_2(t)$, $\rho(t)$, $\theta(t)$ of the system of Eqs. (22). These equations are usually quite complicated. It is at this point that one returns to the approximate solving procedure and applies the straightforward series expansion for the small parameter $\varepsilon$

$$\rho = \rho_0 + \varepsilon\rho_1 + \varepsilon^2\rho_2 + \ldots, \tag{25}$$

$$\theta = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \ldots, \tag{26}$$

$$K_1 = K_{10} + \varepsilon K_{11} + \varepsilon^2 K_{12} + \ldots, \tag{27}$$

$$K_2 = K_{20} + \varepsilon K_{21} + \varepsilon^2 K_{22} + \ldots\ . \tag{28}$$

The relation (27) represents an adiabatic invariant and describes the slow variation of the sector velocity in time due to existence of small functions. The relation (28) is also an adiabatic invariant which considers the energy increase or decrease dependently on the properties of the small forces.

The Taylor series expansion for functions $\Phi_\theta(\rho, \theta, K_1, K_2)$, $\Phi_\rho(\rho, \theta, K_1, K_2)$ and $\mathscr{F}(\rho)$ about $\rho_0$, $\theta_0$, $K_{10}$ and $K_{20}$, which is signed as $()_0$, is

$$\Phi_\theta(\rho, \theta, K_1, K_2) = (\Phi_\theta)_0 + \varepsilon\left[\rho_1\left(\frac{\partial\Phi_\theta}{\partial\rho}\right)_0 + \theta_1\left(\frac{\partial\Phi_\theta}{\partial\theta}\right)_0 + K_{11}\left(\frac{\partial\Phi_\theta}{\partial K_1}\right)_0 + K_{21}\left(\frac{\partial\Phi_\theta}{\partial K_2}\right)_0\right] + \varepsilon^2\ldots, \tag{29}$$

$$\Phi_\rho(\rho, \theta, K_1, K_2) = (\Phi_\rho)_0 + \varepsilon\left[\rho_1\left(\frac{\partial\Phi_\rho}{\partial\rho}\right)_0 + \theta_1\left(\frac{\partial\Phi_\rho}{\partial\theta}\right)_0 + K_{11}\left(\frac{\partial\Phi_\rho}{\partial K_1}\right)_0 + K_{21}\left(\frac{\partial\Phi_\rho}{\partial K_2}\right)_0\right] + \varepsilon^2\ldots, \tag{30}$$

$$\mathscr{F}(\rho) = \int_\rho \rho F(\rho)d\rho = (\mathscr{F})_0 + \varepsilon(\rho_1 + \varepsilon\rho_2 + \ldots)\left(\frac{d\mathscr{F}}{d\rho}\right)_0 + \frac{\varepsilon^2}{2!}(\rho_1 + \varepsilon\rho_2 + \cdots)^2\left(\frac{d^2\mathscr{F}}{d\rho^2}\right)_0 + \cdots. \tag{31}$$

Substituting the functions (25)–(31) into (22) and separating the terms with the same order of the small parameter $\varepsilon$ the following set of equations is obtained:

For $\varepsilon^0$: Eqs. (18)–(21),

$$\text{with initial conditions}: K_{10}(0) = A^2 D, \quad \rho_0(0) = A, \quad \theta_0(0) = C, \tag{32}$$

$$K_{20}(0) = \tfrac{1}{2}(B^2 + c_1 A^2 + A^2 D^2) + \left(\int \rho F(\rho)\mathrm{d}\rho\right)_A,$$

For $\varepsilon$:

$$\begin{aligned}
&\frac{\mathrm{d}K_{11}}{\mathrm{d}t} = (\Phi_\theta)_0 \rho_0,\\
&\frac{\mathrm{d}K_{21}}{\mathrm{d}t} = (\Phi_\rho)_0 \dot\rho_0,\\
&\frac{\mathrm{d}\rho_1}{\mathrm{d}t} = \frac{1}{\dot\rho_0}\left[K_{21} + K_{10}^2\frac{\rho_1}{\rho_0^3} - \frac{K_{10}K_{11}}{\rho_0^2} - \rho_1\left(\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}\rho}\right)_0\right],\\
&\frac{\mathrm{d}\theta_1}{\mathrm{d}t} = \frac{K_{11}}{\rho_0^2} - 2K_{10}\frac{\rho_1}{\rho_0^3},\\
&\text{with initial conditions}: K_{11}(0) = 0, \quad K_{21}(0) = 0,
\end{aligned}\tag{33}$$

$$\rho_1(0) = 0, \quad \theta_1(0) = 0,$$

For $\varepsilon^2$:

$$\begin{aligned}
&\frac{\mathrm{d}K_{12}}{\mathrm{d}t} = (\Phi_\theta)_0\rho_1 + \rho_0\left[\rho_1\left(\frac{\partial\Phi_\theta}{\partial\rho}\right)_0 + \theta_1\left(\frac{\partial\Phi_\theta}{\partial\theta}\right)_0 + K_{11}\left(\frac{\partial\Phi_\theta}{\partial K_1}\right)_0 + K_{21}\left(\frac{\partial\Phi_\theta}{\partial K_2}\right)_0\right],\\
&\frac{\mathrm{d}K_{22}}{\mathrm{d}t} = (\Phi_\rho)_0\dot\rho_1 + \dot\rho_{\varepsilon=0}\left[\rho_1\left(\frac{\partial\Phi_\theta}{\partial\rho}\right)_0 + \theta_1\left(\frac{\partial\Phi_\theta}{\partial\theta}\right)_0 + K_{11}\left(\frac{\partial\Phi_\theta}{\partial K_1}\right)_0 + K_{21}\left(\frac{\partial\Phi_\theta}{\partial K_2}\right)_0\right],\\
&\frac{\mathrm{d}\rho_2}{\mathrm{d}t} = \frac{1}{2\dot\rho_0}\left[2K_{22} + \frac{K_{10}^2}{\rho_0^2}\left(\frac{\rho_1^2}{\rho_0^2} + 2\frac{\rho_2}{\rho_0}\right) + 4\frac{K_{10}K_{11}\rho_1}{\rho_0^3}\right.\\
&\qquad\left. - \frac{K_{11}^2 + 2K_{10}K_{12}}{\rho_0^2} - 2\rho_2\left(\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}\rho}\right)_0 - \rho_1^2\left(\frac{\mathrm{d}^2\mathscr{F}}{\mathrm{d}\rho^2}\right)_0\right] - \frac{\dot\rho_1}{2\dot\rho_0},\\
&\frac{\mathrm{d}\theta_2}{\mathrm{d}t} = \frac{K_{12}}{\rho_0^2} - 2K_{11}\frac{\rho_1}{\rho_0^3} - \frac{K_{10}}{\rho_0^2}\left(\frac{\rho_1^2}{\rho_0^2} + 2\frac{\rho_2}{\rho_0}\right),\\
&\text{with initial conditions}: K_{12}(0) = 0, \quad K_{22}(0) = 0,
\end{aligned}\tag{34}$$

$$\rho_2(0) = 0, \quad \theta_2(0) = 0.$$

$$\cdots$$

For $\varepsilon^0$ the solutions are $\rho_0$, $\theta_0$, $K_{10}$ and $K_{20}$. Substituting these solutions into the set of differential equations (33) with the small parameter $\varepsilon$ the functions $\rho_1$, $\theta_1$, $K_{11}$ and $K_{21}$ are obtained. Namely, integrating the first two relations of (33) for the corresponding initial conditions the functions $K_{11}$ and $K_{21}$ in the first approximation are obtained

$$K_{11}(t) = \varepsilon \int_0^t [\rho \Phi_\theta(\rho, \theta, K_1, K_2)]_0 \mathrm{d}t,$$

$$K_{21}(t) = \varepsilon \int_0^t \left[ \left( 2K_2 - \frac{K_1^2}{\rho^2} + c_1\rho^2 + 2\int \rho F(\rho)\mathrm{d}\rho \right)^{1/2} \Phi_\rho(\rho, \theta, K_1, K_2) \right]_0 \mathrm{d}t. \quad (35)$$

Using the previously obtained functions $\rho_0, \rho_1, \theta_0, \theta_1, K_{10}, K_{11}, K_{20}$ and $K_{21}$ and solving the system of differential equations (34) the functions $\rho_2$, $\theta_2$, $K_{12}$ and $K_{22}$ in the second approximation are obtained.

Usually for technical reasons the solution in the first approximation is satisfactory. The approximate analytic solution in the first approximation and in the complex form is

$$z = (\rho_0 + \varepsilon\rho_1)\exp(\theta_0 + \varepsilon\theta_1). \quad (36)$$

Comparing the suggested procedure with first integrals with the previously mentioned method has an advantage as it gives the solutions of higher order of approximation. The main problem which appears for this method is that it requires the exact solution for the system of differential equations (32) which is very often connected with difficulties.

The procedure with first integrals is very convenient for solving the systems with special initial conditions especially for the case of without impact initial conditions, when the first time derivatives of the polar co-ordinates are zero.

### 3.1. Initial conditions without impact

For this special case of initial conditions

$$\rho(0) = A, \quad \dot{\rho}(0) = 0, \quad \theta(0) = C, \quad \dot{\theta}(0) = 0, \quad (37)$$

i.e.,

$$\rho(0) = A, \quad \theta(0) = C, \quad K_1(0) = 0, \quad K_2(0) = K_{20}, \quad (38)$$

the system of differential equations (32)–(34) simplifies. The corresponding solution of the set of Eqs. (32) for $\varepsilon = 0$ is

$$K_{10} = K_1(0) = 0, \quad \theta_0 = \theta(0) = C,$$

$$K_{20} = K_2(0) = \tfrac{1}{2}c_1A^2 + \left( \int \rho F(\rho)\mathrm{d}\rho \right)_A,$$

$$t = \int_A^{\rho_0} \frac{\mathrm{d}\rho_0}{\sqrt{2K_{20} - c_1\rho_0^2 - 2\mathscr{F}(\rho)_0}}. \quad (39)$$

The functions in the first approximation are

$$K_1 = \varepsilon K_{11}, \quad K_2 = K_{20} + \varepsilon K_{21},$$

$$\theta = C + \varepsilon\theta_1, \quad \rho = \rho_0 + \varepsilon\rho_1, \quad (40)$$

where $K_{11}(t)$, $K_{21}(t)$, $\rho_1(t)$ and $\theta_1(t)$ are the solutions of the system of differential equations

$$\frac{\mathrm{d}K_{11}}{\mathrm{d}t} = (\Phi_\theta)_0\rho_0, \quad \frac{\mathrm{d}K_{21}}{\mathrm{d}t} = (\Phi_\rho)_0\sqrt{2K_{20} - c_1\rho_0^2 - 2\mathscr{F}(\rho)_0},$$
$$\frac{\mathrm{d}\theta_1}{\mathrm{d}t} = \frac{K_{11}}{\rho_0^2}, \quad \frac{\mathrm{d}\rho_1}{\mathrm{d}t} = \frac{1}{\dot\rho_0}\left[K_{21} - c_1\rho_0\rho_1 - \rho_1\left(\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}\rho}\right)_0\right], \tag{41}$$

with $K_{11}(0) = 0$, $K_{21}(0) = 0$, $\rho_1(0) = 0$, $\theta_1(0) = 0$. Eq. $(41)_4$ is the Bernoulli first order ordinary differential equation

$$\frac{\mathrm{d}\rho_1}{\mathrm{d}t} + \rho_1\left[\frac{c_1\rho_0}{\dot\rho_0} + \frac{1}{\dot\rho_0}\left(\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}\rho}\right)_0\right] = \frac{K_{21}}{\dot\rho_0}. \tag{42}$$

Using the complex form of the solution $z = \rho\exp(\mathrm{i}\theta)$ and the previously obtained results it is

$$z = (\rho_0 + \varepsilon\rho_1)\exp(\mathrm{i}(C + \varepsilon\theta_1)). \tag{43}$$

The small function with parameter $\varepsilon$ in the differential equation (15) affects both polar co-ordinates in spite of the fact that the initial conditions are without impact. Dependently on the type of the small non-linear function the following two special cases may appear:

(a) For the case when the right side of the first equation (33) is zero it is $K_{11} = 0$ and $\theta_1 = 0$. The solution in the first approximation is

$$z = (\rho_0 + \varepsilon\rho_1)\exp(\mathrm{i}\theta_0). \tag{44}$$

The small function with $\varepsilon = 0$ has an influence only on the deflection co-ordinate $\rho$, but not on the angle position $\theta$ which remains constant. The solution is periodical along a straight line whose angle is constant and has the value $\theta = \theta(0) = C$. The sectorial velocity is also zero in the first approximation. The small function has an influence only on the energy variation of the system and the amplitude of vibration $\rho$.

(b) For the case when right side of the second equation (33) is zero it is $K_{21} = 0$ and for the first approximation $K_2 = K_{20} = const$. It means that the small function with the parameter $\varepsilon$ has no meaningful influence on energy variation in the first approximation. The deflection co-ordinate $\rho$ does not depend on the small function in the first approximation and it is $\rho = \rho_0$. At the other side this type of small non-linearity introduces a perturbation in the angle position $\theta$ and the sectorial velocity is varying in time. The solution is in the first approximation

$$z = A\exp[\mathrm{i}(C + \varepsilon\theta_1)]. \tag{45}$$

### 3.2. Examples

To prove the correctness of the suggested procedure some examples are considered. For all of them it is common that the strong non-linearity is of cubic type.

#### 3.2.1. The strong non-linearity is cubic

If the strong non-linearity is of cubic type the differential equation (17) transforms to

$$\ddot\rho_0 - \rho_0\dot\theta_0^2 + c_1\rho_0 + \rho_0^3 = 0,$$
$$\rho_0\ddot\theta_0 + 2\dot\rho_0\dot\theta_0 = 0, \tag{46}$$

with initial conditions

$$\rho(0) = A, \quad \dot{\rho}(0) = 0, \quad \theta(0) = C, \quad \dot{\theta}(0) = 0. \tag{47}$$

For the system (46) with initial conditions (47) according to (39) the first integrals are

$$K_{10} = K_1(0) = 0,$$

$$K_{20} = K_2(0) = \tfrac{1}{2} c_1 A^2 + \tfrac{1}{4} c_3 A^4. \tag{48}$$

The polar co-ordinates are

$$\rho_0 = A cn(\omega t, m), \quad \theta_0 = \theta(0) = C = const., \tag{49}$$

where

$$m = \frac{c_3 A^2}{2(c_1 + c_3 A^2)}, \quad \omega = \sqrt{c_1 + c_3 A^2}. \tag{50}$$

$cn(\omega t, m)$ is a Jacobi elliptic function [34] where $m$ is the modulus of the function and $\omega t$ its argument. Using the relations (49) the solution of (46) in the complex form is

$$z = A cn(\omega t, m) \exp(iC). \tag{51}$$

This solution is the exact closed form analytical solution for the initial conditions (47). The solution is periodical and it is along a straight line with constant inclination angle $\theta_0 = \theta(0) = C$.

For the numerical values $c_1 = 2$ and $c_3 = 1$ and initial conditions

$$\rho(0) = A = 0.1, \quad \dot{\rho}(0) = 0, \quad \theta(0) = C = \pi/3, \quad \dot{\theta}(0) = 0, \tag{52}$$

it is

$$z = 0.1 cn(1.41774t, 0.00284) \exp(i\pi/3), \tag{53}$$

where

$$m = 0.00284, \quad \omega = 1.41774. \tag{54}$$

### 3.2.2. *Small gyroscopic function*

For the small gyroscopic function the differential equation is

$$\ddot{z} + c_1 z + c_3 z(z\bar{z}) = 2\varepsilon g i \dot{z}(1 + p(z\bar{z})), \tag{55}$$

where $\varepsilon g$ is the coefficient of the gyroscopic force and $p$ is a small coefficient of non-linearity. Transforming the small function on the right side into polar co-ordinates it is

$$\Phi_\rho = -2\varepsilon g \rho \dot{\theta}(1 + p\rho^2), \quad \Phi_\theta = 2\varepsilon g \dot{\rho}(1 + p\rho^2). \tag{56}$$

Due to (41) it is evident that as $(\Phi_\rho)_0 = 0$ it is $K_{21} = 0$ and $\rho_1 = 0$. The functions $K_{11}$ and $\theta_1$ are

$$K_{11} = -\varepsilon g A^2\left(1 + \frac{pA^2}{2}\right) + \varepsilon g A^2 cn^2(\omega t, m)\left(1 + \frac{pA^2}{2} cn^2(\omega t, m)\right), \tag{57}$$

$$\theta_1 = \varepsilon g\left[\frac{E(\omega t, m)}{\omega m'}\left(1 + \frac{pA^2}{2m}\right) - \frac{1}{\omega m'}\left(1 + \frac{pA^2}{2}\right) dn(\omega t, m) tn(\omega t, m) - \frac{pA^2}{2m} t\right], \tag{58}$$

where $m' = 1 - m$, $E(\omega t, m)$ is the Legendre's incomplete elliptic integral of the second kind, $dn$ and $tn$ are Jacobi elliptic functions [34] with modulus $m$ and frequency $\omega$ (48). The solution in the first approximation is

$$z = Acn(\omega t, m)\exp\left\{\mathrm{i}C + \mathrm{i}\varepsilon g\left[\frac{E(\omega t, m)}{\omega m'}\left(1 + \frac{pA^2}{2m}\right) - \frac{1}{\omega m'}\left(1 + \frac{pA^2}{2}\right)dn(\omega t, m)tn(\omega t, m) - \frac{pA^2}{2m}t\right]\right\}. \tag{59}$$

The solution is periodical and two different periods appear: $\omega$ and $\Omega$ which correspond to Jacobi elliptic function and the circular function, respectively. For $\omega \approx \Omega$ the effect of fluttering is evident.

For the numerical values $c_1 = 2$, $c_3 = 1$, $\varepsilon g = 0.1$ and $p = 1$ and initial conditions (52) it is

$$z = 0.1cn(1,41774t, 0.00284) \times \exp\left\{\mathrm{i}\left[\frac{\pi}{3} + 0.19526E(1.41774t, 0.00284) - 0.17857t - 0.071089dn(1,41774t, 0.00284)tn(1,41774t, 0.00284)\right]\right\}. \tag{60}$$

Using the fact that the modulus $m$ (54) has the value approximately zero and after expansion in series the elliptic functions and elliptic integral $E$ (see Ref. [35]) it is

$$z \approx 0.1\cos(1.41774t)\exp\left[\mathrm{i}\left(\frac{\pi}{3} + 0.09826t\right)\right]. \tag{61}$$

To prove the correctness of the approximate analytical solution (61) it is compared with the exact numerical one. For the introduced numerical values the differential equation (55) is

$$\ddot{x} + 2x + x(x^2 + y^2) = -2\varepsilon g\dot{y}(1 + p(x^2 + y^2)),$$

$$\ddot{y} + 2y + y(x^2 + y^2) = 2\varepsilon g\dot{x}(1 + p(x^2 + y^2)), \tag{62}$$

where the initial conditions (52) are

$$x(0) = 0.05, \quad \dot{x}(0) = 0, \quad y(0) = 0.0866, \quad \dot{y}(0) = 0. \tag{63}$$

Applying the Runge Kutta numerical procedure the exact solution of the system (62) is obtained. In Fig. 1 the analytical ($x_A - t$ and $y_A - t$) (61) and numerical solution ($x_N - t$ and $y_N - t$) are compared. It is evident that the analytical solution is on top of the numerical one. The difference is negligible for a long time period and also for the small value to $\varepsilon = 0.1$.

### 3.2.3. Small damping function

The differential equation with small damping function is

$$\ddot{z} + c_1 z + c_3 z(z\bar{z}) = -\varepsilon b\dot{z}, \tag{64}$$

where $\varepsilon b$ is the coefficient of the damping function. Transforming the small function on the right side into polar co-ordinates it is

$$\Phi_\rho = -\varepsilon b\dot{\rho}, \quad \Phi_\theta = -\varepsilon b\rho\dot{\theta}. \tag{65}$$

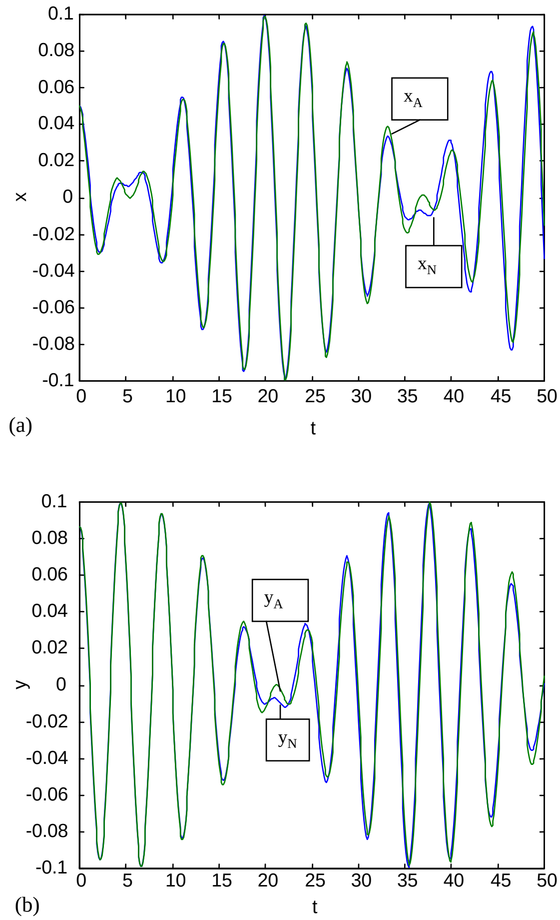


Fig. 1. Time–history diagrams when the gyroscopic term exists: (a) $x - t$ obtained analytical ($x_A - t$) and numerical ($x_N - t$), and (b) $y - t$ obtained analytical ($y_A - t$) and numerical ($y_N - t$).

Due to (41) it is evident that as $(\Phi_\theta)_0 = 0$ it is $K_{11} = 0$ and $\theta_1 = 0$. The function $K_{21}$ is

$$K_{21} = -\varepsilon b A^2 \omega \left[ \frac{1-m}{3m} \omega t + \frac{2m-1}{3m} E(\omega t, m) - \tfrac{1}{3} sn(\omega t, m) cn(\omega t, m) dn(\omega t, m) \right]. \tag{66}$$

According to (42) the polar co-ordinate $\rho_1$ is obtained from the Bernoulli equation

$$\frac{\mathrm{d}\rho_1}{\mathrm{d}t} + f(t)\rho_1 = g(t), \tag{67}$$

where

$$f(t) = -\frac{(c_1 + c_3 A^2 cn^2(\omega t, m)) cn(\omega t, m)}{\omega sn(\omega t, m) dn(\omega t, m)},$$

$$g(t) = \varepsilon b A \left( \frac{(1-m)\omega t + (2m-1)E(\omega t, m)}{3m sn(\omega t, m) dn(\omega t, m)} - \tfrac{1}{3} cn(\omega t, m) \right). \tag{68}$$

For the parameter values $\varepsilon b = 0.1$, $c_1 = 2$ and $c_3 = 1$ and initial conditions (52) the modulus of the Jacobi elliptic function (54) is approximately zero and the function (66) simplifies to

$$K_{21} \approx -\frac{\varepsilon b A^2 \omega}{2}\left(\omega t - \frac{\sin(2\omega t)}{2}\right). \tag{69}$$

The approximate solution of the Bernoulli differential equation (67) for (69) after some transformation and simplification is

$$\rho_1 \approx -\frac{\varepsilon b}{2\omega} A t \cos(\omega t). \tag{70}$$

For the aforementioned numerical values the approximate analytic solution of (64) is

$$z \approx 0.1(1 - 0.035t)\cos(1.41774t)\exp(\mathrm{j}\pi/3). \tag{71}$$

The differential equation (64) is solved numerically. Applying the Runge–Kutta numerical procedure the exact numerical solution is obtained. In Fig. 2 the approximate analytical solution

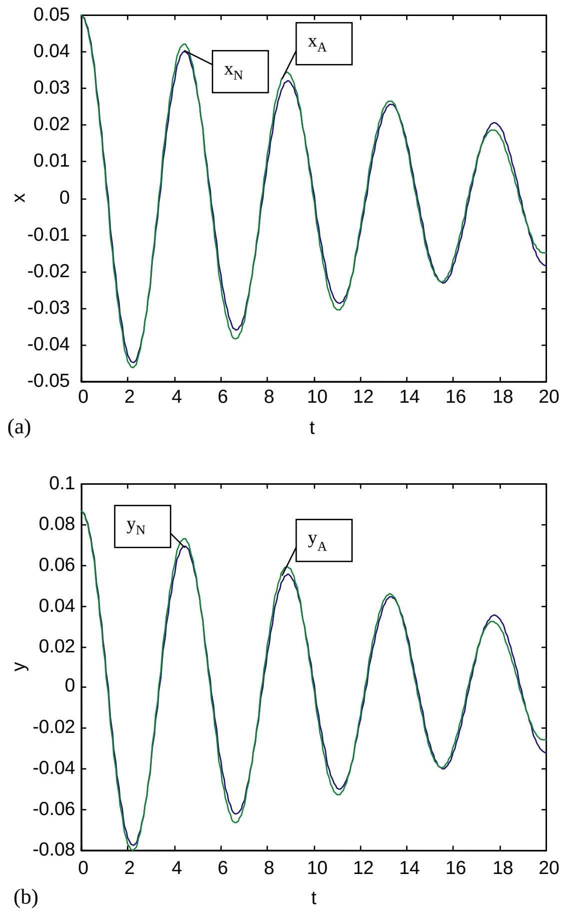


Fig. 2. Time–history diagrams when the damping term exists: (a) $x - t$ obtained analytical ($x_A - t$) and numerical ($x_N - t$), and (b) $y - t$ obtained analytical ($y_A - t$) and numerical ($y_N - t$).

($x_A - t$ and $y_A - t$) (71) is compared with numerical ($x_N - t$ and $y_N - t$) one. It can be concluded that the analytical solution is on top of the numerical solution. From the solution it can be seen that the small damping function which is a linear function of the time derivative of the complex function $z$ decreases the amplitude $\rho$ in time and tends to zero. The velocity of amplitude decrease depends not only on the value of coefficient of damping $\varepsilon b$ but also on the value of the initial amplitude $A$ which affects the frequency of vibration $\omega$.

## 4. Conclusion

From the previous considerations it can be concluded:

1. The generalized Krylov–Bogolubov solution procedure for the differential equation (1) with complex function is a very convenient one due to its simplicity. The approximate analytical solution obtained in the first approximation is quite satisfactory for technical reasons for the small value of the parameter $\varepsilon$ and short time interval. The disadvantage of the method is that it is applicable if the generating closed form analytical solution is known.
2. The approximate analytic method based on the first integrals is especially appropriate for the special initial conditions where the initial conditions which are the first time derivatives of the polar co-ordinates are zero. Those initial conditions are named 'initial conditions without impact'. Using the straightforward series expansion the first and higher order approximate analytical solution is obtained. Comparing the analytical approximate solution with the exact numerical one it is concluded that the difference is negligible.
3. The mathematical model (1) considered in this paper represents the differential equation of the vibration of a strong non-linear Jeffcott rotor (symmetrically supported shaft-disc system) with small forces. The strong non-linearity is usually caused by the non-linear elastic force in the shaft. The analytical solutions obtained for the system with strong cubic non-linearity, small gyroscopic force and small damping force considered in this paper give very useful information about the vibrations of the rotor and are of special interest for rotor dynamics.

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